Strong regularization by noise for a class of kinetic SDEs

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Introduction: kinetic SDEs driven by a stable process

SDE in $\mathbb{R}^{2d}$

$$\begin{cases} dV_t = F_1(t, V_t, X_t)dt + \sigma(t, V_t, X_t)dL_t^{(\alpha)} \\ dX_t = \big(V_t + F_2(t, X_t)\big)\, dt \end{cases}$$

Applications:

- particles' motion (*Villani*, 2002)
- hydrogeology (*Zhang, Y. & others*, 2017)
- mathematical finance (*Pascucci*, 2011)
- others... (*Delarue-Menozzi*, 2010)

Main question: strong well-posedness for low regularity coefficients

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State of the art: continuous noise

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Coefficients' regularity:

$$|F_1(t, v, x) - F_1(t, v', x')| \lesssim |v - v'|^{\beta_1^1} + |x - x'|^{\beta_1^2}$$
$$|F_2(t, x) - F_2(t, x')| \lesssim |x - x'|^{\beta_2}$$

- *Strong uniqueness* (*Chaudru De Raynal-Honoré-Menozzi*, 2022):

$$\begin{matrix} \beta_1^1 \in]0, 1] \\ \beta_1^2, \beta_2 \in]2/3, 1] \end{matrix} \implies \textit{strong uniqueness}$$

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Counterexample: $F_2(x) = \text{sign}(x)|x|^{\beta_2}$ $\quad \beta_2 \in]0, 1/3[$

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Non-degenerate regularization by noise

$$dX_t = F(t, X_t)dt + dW_t \qquad X_t = X_0 + \int_0^t F(s, X_s)ds + W_t$$

Zvonkin transform:

$$\begin{cases} \frac{1}{2}\partial_{xx}u(t,x) + F(t,x)\partial_x u(t,x) + \partial_t u(t,x) = -F(t,x) \\ u(T,\cdot) = 0 \end{cases}$$

$$\xrightarrow[\text{formula}]{\text{Itô}} du(t, X_t) = -F(t, X_t)dt + \partial_x u(t, X_t)dW_t$$

$$\Longrightarrow \int_0^t F(s, X_s)ds = u(0, X_0) - u(t, X_t) + \int_0^t \partial_x u(s, X_s)dW_s$$

Crucial point: u and $\partial_x u$ are Lipschitz continuous

Zvonkin (1974), Veretennikov (1981), Flandoli (2015)

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Kinetic regularization by noise

Degenerate settings:

$$\begin{cases} dV_t = dW_t \\ dX_t = F(t, X_t)dt + V_t dt \end{cases} \qquad t \in [0, T]$$

Non-Lipschitz-continuous coefficient:

$$|F(t,x) - F(t,x')| \leq C|x - x'|^\beta$$

$\beta \in (0,1]$ s.t. strong uniqueness for SDE?

Backward Kolmogorov operator:

$$\mathcal{L} = \frac{1}{2}\partial_{vv} + (v + F(t,x))\partial_x + \partial_t \qquad (t,x,v) \in [0,T] \times \mathbb{R}^2$$

Degenerate PDE $\implies$ $\nabla_{(x,v)}u$ Lipschitz continuous in x?

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Anisotropic scaling and smoothing

Langevin prototype model in $\mathbb{R}^2$:

$$\begin{cases} dV_t = dW_t \\ dX_t = V_t\, dt \end{cases}$$

Scaling:

$$\left.\begin{array}{l} \mathbb{E}[V_t^2] = \mathbb{E}[W_t^2] = t \\ \mathbb{E}[X_t^2] = \mathbb{E}\left[\left(\int_0^t V_s ds\right)^2\right] \leq t\int_0^t \mathbb{E}[V_s^2]ds \approx t^3 \end{array}\right\} \implies V_t \approx X_t^{1/3}$$

Backward Kolmogorov operator:

$$\mathcal{L} = \frac{1}{2}\partial_{vv} + v\partial_x + \partial_t$$

$$\mathcal{L}u = F \in C_x^{\beta/3} C_v^{\beta} \implies \begin{cases} u & \in C_x^{\frac{2+\beta}{3}} C_v^{2+\beta} \\ \nabla_v u & \in C_x^{\frac{1+\beta}{3}} C_v^{1+\beta} \end{cases}$$

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Main result: strong well-posedness in the Brownian case

SDE in $[0, T] \times \mathbb{R}^{2d}$
$$\begin{cases} dV_t = F_1(t, V_t)dt + \sigma(t, V_t)dW_t \\ dX_t = V_t + F_2(t, X_t)dt \end{cases}$$

Ellipticity on $\mathbb{R}^d$: $\quad \Lambda^{-1}|\eta|^2 \le \langle \sigma(t, v)\eta, \eta\rangle \le \Lambda|\eta|^2$

Coefficients' regularity:

$$\begin{aligned} |\sigma(t, v) - \sigma(t, v')| &\lesssim |v - v'| \\ |F_1(t, v) - F_1(t, v')| &\lesssim |v - v'|^{\beta_1} \\ |F_2(t, x) - F_2(t, x')| &\lesssim |x - x'|^{\beta_2} \end{aligned}$$

- $\sigma, F_1, F_2(\cdot, 0)$ bounded
- $\beta_1 \in]0, 1],\ \beta_2 \in]1/3, 1]$
- *Gradient estimate* for F_2: $\ \forall 0 \le t < s \le T,\ \theta \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} |\nabla F_2(s, y)|\Gamma((s-t)^3, y-\theta)dy \le C_T(s-t)^{\frac{3}{2}(\beta_2 - 1)}$$

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Examples and counterexamples

Chaudru de Raynal-Menozzi-Pesce-Zhang (2023):

$$\int_t^T \mathbb{E}[|\nabla_x F^{(\varepsilon)}(s, X_s^{\varepsilon,t,v,x})|]ds \leq C_T, \qquad (t,v,x) \in [0,T] \times \mathbb{R}^{2d}, \quad \varepsilon > 0$$

Example: Peano type functions

$$F(t,x) := a(t)|x|^{\beta_2} \qquad \beta_2 \in]1/3, 1] \qquad a \in L^\infty$$

Counterexample: cumulating singularities (d=1)

$$F(x) = \sum_{n \in \mathbb{N}} F_n(x) \mathbb{1}_{[a_n, a_{n+1}[}(x) \qquad \beta \in [0,1]$$

$$F_n(x) = \begin{cases} (x - a_n)^\beta & \text{if } x \in \left[a_n, \frac{a_n + a_{n+1}}{2}\right[\\ (a_{n+1} - x)^\beta & \text{if } x \in \left[\frac{a_n + a_{n+1}}{2}, a_{n+1}\right[\end{cases}$$

$$a_0 = 0, \; a_n \nearrow 1 \qquad a_{n+1} - a_n \approx n^{-1/\beta}$$

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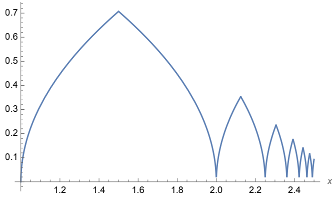

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First order Zvonkin transform technique (1)

Autonomous equation:

$$dV_t = F_1(t, V_t)dt + \sigma(t, V_t)dW_t$$

Random transport equation on $]0, \tau[\times\mathbb{R}^d$:

$$\begin{cases} \partial_t u_\varepsilon(t,x) + \nabla_x u_\varepsilon(t,x)\big(V_t(\omega) + F^{(\varepsilon)}(t,x)\big) = F^{(\varepsilon)}(t,x) \\ u_\varepsilon(\tau,\cdot) \equiv 0 \end{cases}$$

$\implies$ solution:

$$u_\varepsilon(t,x) = -\int_t^\tau g_\varepsilon(s,x)ds$$

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First order Zvonkin transform technique (2)

Deterministic Itô formula:

$$u_\varepsilon\big(0, X_0(\omega)\big)$$
$$= -\int_0^\tau g_\varepsilon(t, X_t(\omega))dt - \int_0^\tau \nabla_x u_\varepsilon(t, X_t(\omega))\big(V_t(\omega) + F(t, X_t(\omega))\big)dt$$

$$\implies \int_0^\tau F^{(\varepsilon)}(t, X_t)dt = \int_0^\tau \Big(g_\varepsilon(t, X_t) + \nabla_x u_\varepsilon(t, X_t)\big(V_t + F^{(\varepsilon)}(t, X_t)\big)\Big)dt$$

$$= \int_0^\tau \nabla_x u_\varepsilon(t, X_t)\big(F^{(\varepsilon)}(t, X_t) - F(t, X_t)\big)dt$$
$$- u_\varepsilon(0, X_0)$$

First order Zvonkin transform technique (2)

Deterministic Itô formula:

$$
\begin{aligned}
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\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \int_0^\tau F^{(\varepsilon)}(t, X_t)dt &= \int_0^\tau \Big(g_\varepsilon(t, X_t) + \nabla_x u_\varepsilon(t, X_t)\big(V_t + F^{(\varepsilon)}(t, X_t)\big)\Big)dt \\
&= \int_0^\tau \nabla_x u_\varepsilon(t, X_t)\big(F^{(\varepsilon)}(t, X_t) - F(t, X_t)\big)dt \\
&\quad - u_\varepsilon(0, X_0)
\end{aligned}
$$

First order Zvonkin transform technique (3)

$(X_t)_{t\in[0,T]}, (X'_t)_{t\in[0,T]}$ solutions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$
$X_0 = X'_0$ $\mathbb{P}$-almost surely.

$$
\begin{aligned}
X_\tau - X'_\tau &= \int_0^\tau F(t, X_t)dt - \int_0^\tau F(t, X'_t)dt \\
&= \int_0^\tau \nabla_x u_\varepsilon(t, X_t)\big(F^{(\varepsilon)}(t, X_t) - F(t, X_t)\big)dt \\
&\quad + \int_0^\tau \big(F(t, X_t) - F^{(\varepsilon)}(t, X_t)\big)dt \\
&\quad + \int_0^\tau \nabla_x u_\varepsilon(t, X'_t)\big(F^{(\varepsilon)}(t, X'_t) - F(t, X'_t)\big)dt \\
&\quad + \int_0^\tau \big(F(t, X'_t) - F^{(\varepsilon)}(t, X'_t)\big)dt
\end{aligned}
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First order Zvonkin transform technique (4)

By Hölder inequality: $(p^{-1} + q^{-1} = 1)$

$$\mathbb{E}\big[|X_\tau - X'_\tau|\big] \le \int_0^\tau \mathbb{E}\big[|\nabla_x u_\varepsilon(t, X_t)|^q\big]^{\frac{1}{q}} \times \mathbb{E}[|F^{(\varepsilon)}(t, X_t) - F(t, X_t)|^p]^{\frac{1}{p}} dt$$

$$+ \int_0^\tau \mathbb{E}\big[|\nabla_x u_\varepsilon(t, X'_t)|^q\big]^{\frac{1}{q}} \times \mathbb{E}[|F^{(\varepsilon)}(t, X'_t) - F(t, X'_t)|^p]^{\frac{1}{p}} dt$$

$$+ \int_0^\tau \mathbb{E}[|F^{(\varepsilon)}(t, X_t) - F(t, X_t)|]dt$$
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Density estimate for the ODE

$$\mathbb{E}\big[|\nabla_x u_\varepsilon(t, X_t)|^q\big] + \mathbb{E}\big[|\nabla_x u_\varepsilon(t, X'_t)|^q\big] \le C_{T,q}, \qquad t \in [0, \tau], \quad \varepsilon > 0$$

Main result: strong well-posedness in 2-dimensional setting

SDE in $[0,T] \times \mathbb{R}^2$ $$\begin{cases} dV_t = \mu(t,V_t)dt + \sigma(t,V_t)dL_t^{(\alpha)} \\ dX_t = V_t + F(t,X_t)dt \end{cases}$$

Coefficients' regularity

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Recall: state of the art

Kinetic-type equation:

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[Chaudru-Menozzi, 2022] [Chaudru-Honoré-Menozzi, 2022]

$\beta \in]1/3, 1] \implies$ *weak uniqueness*

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Watanabe-Yamada technique:

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Schauder estimates for degenerate equations

Backward Kolmogorov operator:

$$\mathcal{L} := \frac{1}{2}\partial_{vv} + (v + F(t,x))\partial_x + \partial_t$$

Cauchy problem:

$$\begin{cases} \mathcal{L}u = F & \text{on } [0,T] \times \mathbb{R}^2 \\ u(T,\cdot) = 0 \end{cases}$$

[Hao-Wu-Zhang, 2020]

$$\beta > 1/3 \quad F(t,\cdot) \in C^\beta \quad F(\cdot,0) \in L^\infty \quad \Longrightarrow \quad \begin{aligned} u \in\ & L^\infty_T C^1_x C^2_v \\ \partial_v u \in\ & L^\infty_T C^{1/3+\beta}_x C^1_v \end{aligned}$$

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Introducing stable processes

$L_t^{(\alpha)}$ stable process of index $\alpha \in]1, 2[$

Symmetric Lévy process generated by the fractional Laplacian

$$\Delta_v^{\alpha/2} f(v) = \int_{\mathbb{R}} \big(f(t, v + w) - f(t, v - w) - 2f(t, v) \big) \frac{dw}{|w|^{1+\alpha}}$$

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